

Mathematics

Common Core State Standards

Old Boxes

- People are the next step
- If people just swap out the old standards and put the new CCSS in the old boxes
 - into old systems and procedures
 - into the old relationships
 - Into old instructional materials formats
 - Into old assessment tools,
- Then nothing will change, and perhaps nothing will

Standards are a platform for instructional systems

This is a new platform for better instructional
systems and better ways of managing
instruction

Builds on achievements of last 2 decades

Builds on lessons learned in last 2 decades

Lessons about time and teachers

Mathematical Practices Standards

1. Make sense of complex problems and persevere in solving them.
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning.

College and Career Readiness Standards for Mathematics

Expertise and Character

- Development of expertise from novice to apprentice to expert
 - Schoolwide enterprise: school leadership
 - Department wide enterprise: department taking responsibility
- The Content of their mathematical Character
 - Develop character

Grain size is a major issue

- Mathematics is simplest at the right grain size.
- “Strands” are too big, vague e.g. “number”
- Lessons are too small: too many small pieces scattered over the floor, what if some are missing or broken?
- Units or chapters are about the right size (8-12 per year)
- STOP managing lessons,
- START managing units

What mathematics do we want students to walk away with from this chapter?

- Content Focus of professional learning communities should be at the chapter level
- When working with standards, focus on clusters. Standards are ingredients of clusters. Coherence exists at the cluster level across grades
- Each lesson within a chapter or unit has the same objectives....the chapter objectives

Each chapter

- Teach diagnostically early in the unit:
 - What mathematics are my students bringing to this chapter's mathematics
 - Take a problem from end of chapter
 - Tells you which lessons need dwelling on, which can be fast
- Converge students on the chapter mathematics later in the unit
 - Pair students to optimize tutoring and development of proficiency in explaining mathematics

Teachers should manage lessons

- Lessons take one or two days or more depending on how students respond
- Yes, pay attention to how they respond
- Each lesson in the unit has the same learning target which is a cluster of standards
- “what mathematics do I want my students to walk away with from this chapter?”

Social Justice

- Main motive for standards
- Get good curriculum to all students
- Start each unit with the variety of thinking and knowledge students bring to it
- Close each unit with on-grade learning in the cluster of standards

Mathematics Standards Design

Common Core State Standards

COMMON CORE STATE STANDARDS FOR

Mathematics



Two major design principles, based on evidence:

- **Focus**
- **Coherence**

The Importance of Focus

- TIMSS and other international comparisons suggest that the U.S. curriculum is ‘a mile wide and an inch deep.’
- “On average, the U.S. curriculum omits only 17 percent of the TIMSS grade 4 topics compared with an average omission rate of 40 percent for the 11 comparison countries. The United States covers all but 2 percent of the TIMSS topics through grade 8 compared with a 25 percent non coverage rate in the other countries. **High-scoring Hong Kong’s curriculum omits 48 percent of the TIMSS items through grade 4, and 18 percent through grade 8. Less topic coverage can be associated with higher scores on those topics covered because students have more time to master the content that is taught.”**
- Ginsburg et al., 2005

U.S. standards organization

[Grade 1]

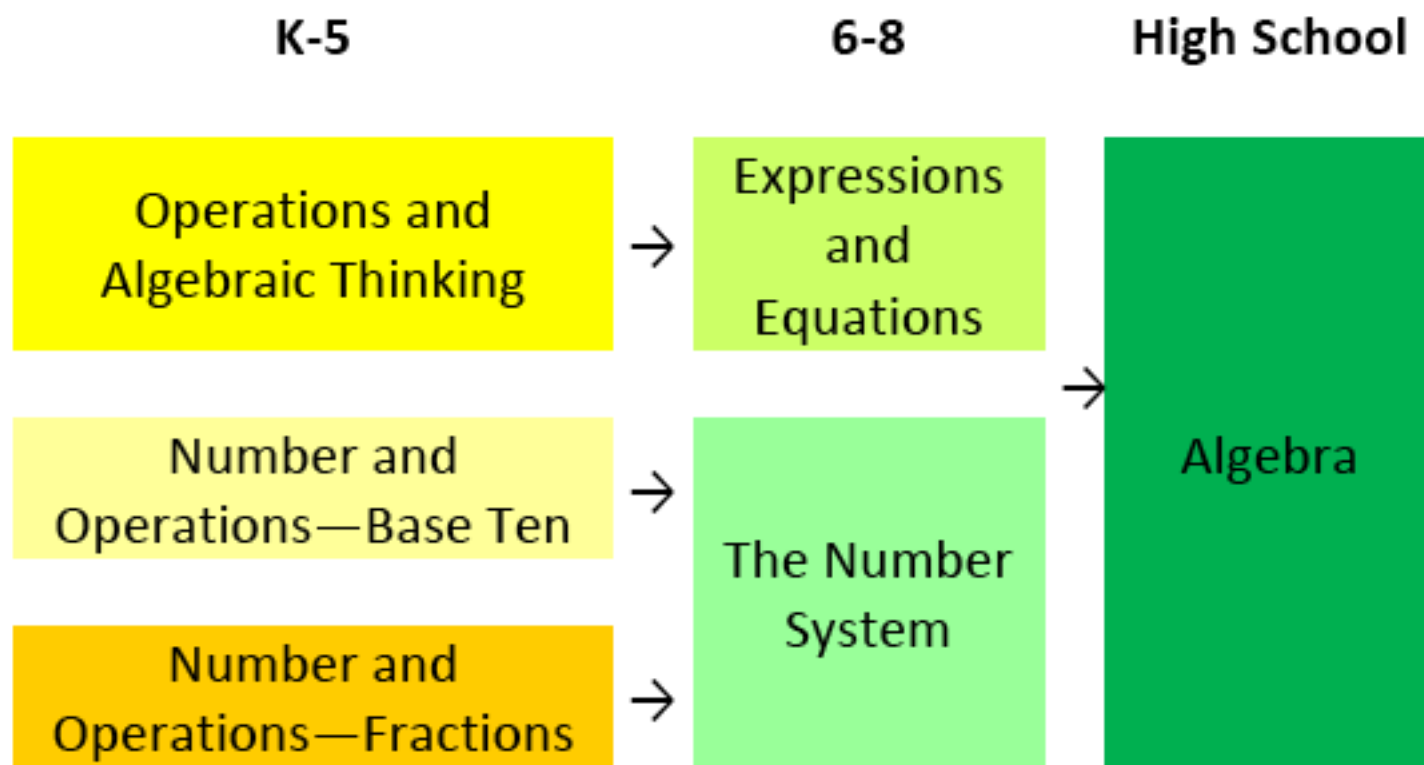
- Number and Operations
 - ...
- Measurement and Geometry
 - ...
- Algebra and Functions
 - ...
- Statistics and Probability
 - ...

U.S. standards organization

[12]

- Number and Operations
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Focusing attention within Number and Operations



The most important ideas in the CCSS mathematics that need attention.

1. Properties of operations: their role in arithmetic and algebra
2. Mental math and [algebra vs. algorithms]
3. Units and unitizing
4. Operations and the problems they solve
5. Quantities-variables-functions-modeling
6. Number-Operations-Expressions-Equation
7. Modeling
8. Practices

Mental math

$$72 - 29 = ?$$

In your head.

Composing and decomposing

Partial products

Place value in base 10

Factor $X^2 + 4x + 4$ in your head

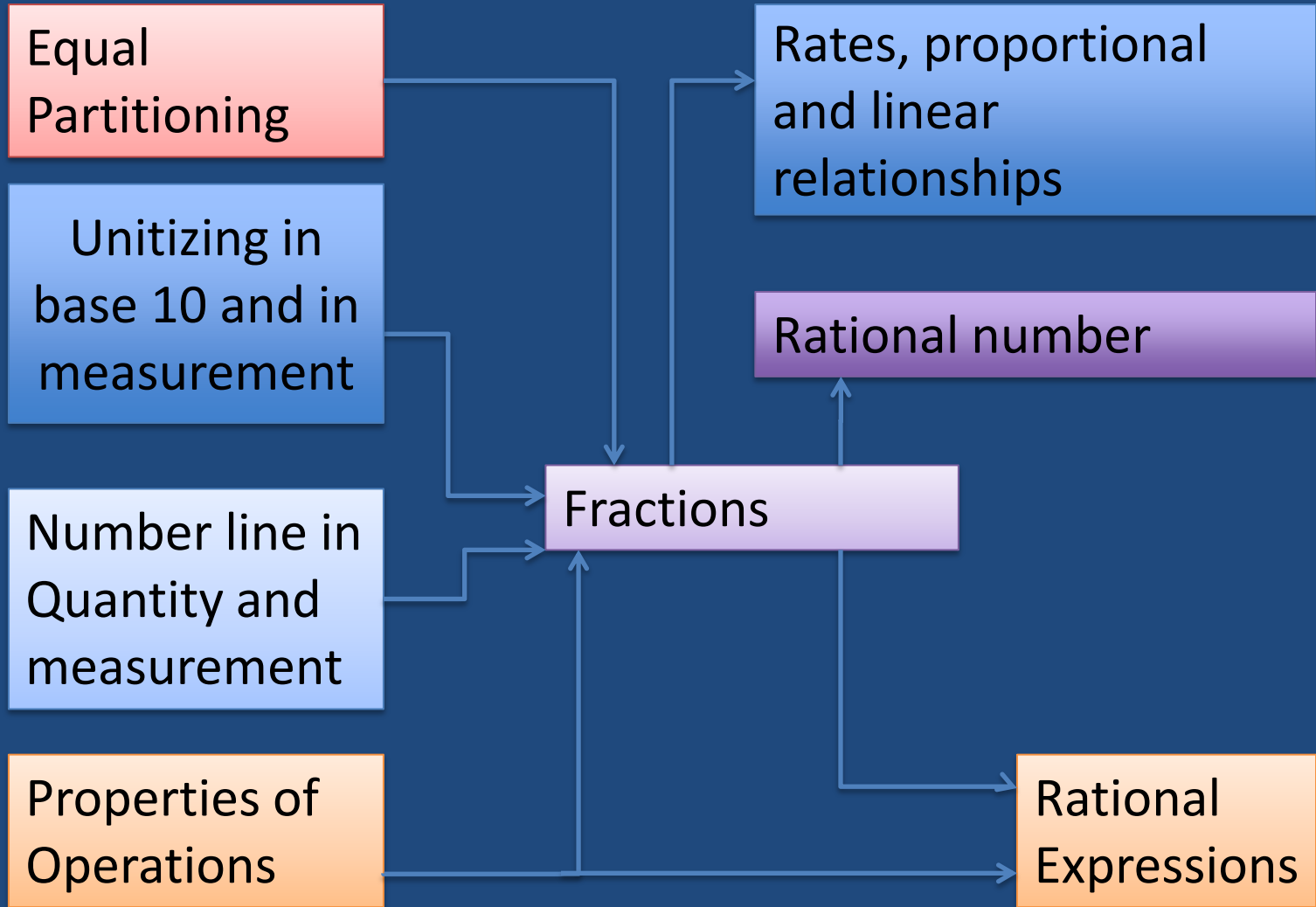
Fractions Progression

- Understanding the arithmetic of fractions draws upon four prior progressions that informed the CCSS:
 - equal partitioning,
 - unitizing,
 - number line,
 - and operations.

K - 2

3 - 6

7 - 12



Partitioning

- The first two progressions, equal partitioning and unitizing, draw heavily from learning trajectory research. Confrey has established how children develop ideas of partitioning from early experiences with fair sharing and distributing. These developments have a life of their own apart from developing counting and adding

Unitizing

- . Clements and also Steffe have established the importance of children being able to see a group(s) of objects or an abstraction like 'tens' as a unit(s) that can be counted.
- Whatever can be counted can be added, and from there knowledge and expertise in whole number arithmetic can be applied to newly unitized objects, like counting tens in base 10, or adding standard lengths such as inches in measurement.

Units are things you count

- Objects
- Groups of objects
- 1
- 10
- 100
- $\frac{1}{4}$ unit fractions
- Numbers represented as expressions

Units add up

- 3 pennies + 5 pennies = 8 pennies
- 3 ones + 5 ones = 8 ones
- 3 tens + 5 tens = 8 tens
- 3 inches + 5 inches = 8 inches
- 3 $\frac{1}{4}$ inches + 5 $\frac{1}{4}$ inches = 8 $\frac{1}{4}$ inches
- $\frac{3}{4} + \frac{5}{4} = \frac{8}{4}$
- $3(x + 1) + 5(x+1) = 8(x+1)$

Unitizing links fractions to whole number arithmetic

- Students' expertise in whole number arithmetic is the most reliable expertise they have in mathematics
- It makes sense to students
- If we can connect difficult topics like fractions and algebraic expressions to whole number arithmetic, these difficult topics can have a solid foundation for students

Operations and the problems they solve

- Tables 1 and 2 on pages 88 and 89

From table 2 page 89

- $a \times b = ?$
 - $a \times ? = p$, and $p \div a = ?$
 - $? \times b = p$, and $p \div b = ?$
-
- *1. Play with these using whole numbers,*
 - *2. make up a problem for each.*
 - *3. substitute $(x - 1)$ for b*

“Properties of Operations”

- Also called “rules of arithmetic” , “number properties”

Nine properties are the most important preparation for algebra

- Just nine: foundation for arithmetic
- Exact same properties work for whole numbers, fractions, negative numbers, rational numbers, letters, expressions.
- Same properties in 3rd grade and in calculus
- Not just learning them, but learning to use them

Using the properties

- To express yourself mathematically (formulate mathematical expressions that mean what you want them to mean)
- To change the form of an expression so it is easier to make sense of it
- To solve problems
- To justify and prove

Properties are like rules, but also like rights

- You are allowed to use them whenever you want, never wrong.
- You are allowed to use them in any order
- Use them with a mathematical purpose

Properties of addition

| | |
|---|--|
| <i>Associative property of addition</i> | $(a + b) + c = a + (b + c)$ $(2 + 3) + 4 = 2 + (3 + 4)$ |
| <i>Commutative property of addition</i> | $a + b = b + a$ $2 + 3 = 3 + 2$ |
| <i>Additive identity property of 0</i> | $a + 0 = 0 + a = a$ $3 + 0 = 0 + 3 = 3$ |
| <i>Existence of additive inverses</i> | For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0.$ $2 + (-2) = (-2) + 2 = 0$ |

Properties of multiplication

| | |
|---|---|
| <i>Associative property of multiplication</i> | $(a \times b) \times c = a \times (b \times c)$ $(2 \times 3) \times 4 = 2 \times (3 \times 4)$ |
| <i>Commutative property of multiplication</i> | $a \times b = b \times a$ $2 \times 3 = 3 \times 2$ |
| <i>Multiplicative identity property of 1</i> | $a \times 1 = 1 \times a = a$ $3 \times 1 = 1 \times 3 = 3$ |
| <i>Existence of multiplicative inverses</i> | For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$ $2 \times 1/2 = 1/2 \times 2 = 1$ |

Linking multiplication and addition: the ninth property

- *Distributive property of multiplication over addition*

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$a(b+c) = ab + ac$$

Find the properties in the multiplication table

- There are many patterns in the multiplication table, most of them are consequences of the properties of operations:
- Find patterns and explain how they come from the properties.
- Find the distributive property patterns

Grade level examples

- 3 packs of soap
- 4 dealing cards
- 5 sharing
- 6 money
- 7 lengths (fractions)
- 8 times larger (%)

K -5

Quantity and measurement

Operations and algebraic thinking

Modeling Practices

6 - 8

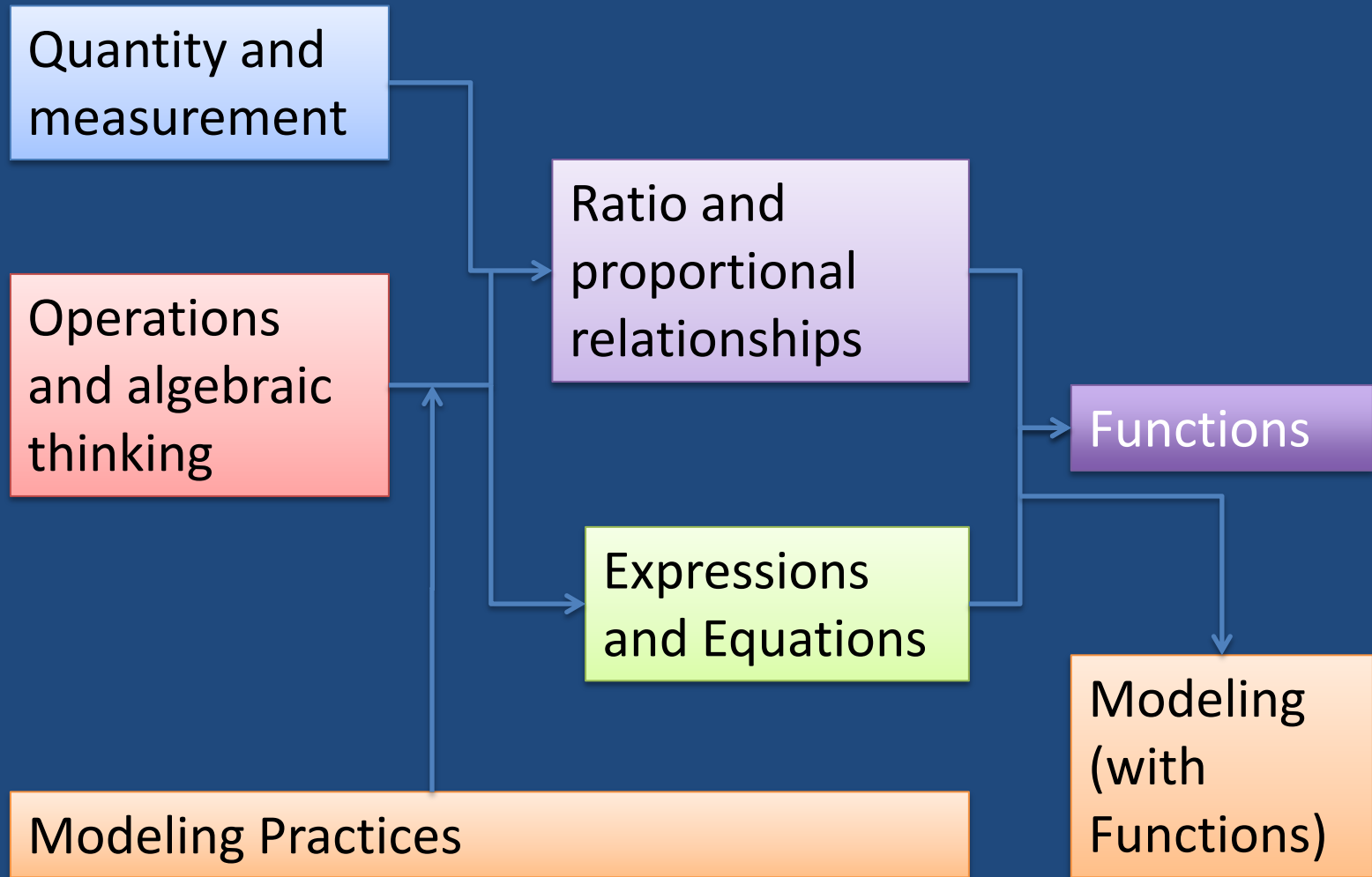
Ratio and proportional relationships

Expressions and Equations

9 - 12

Functions

Modeling (with Functions)



Functions and Solving Equations

1. Quantities-variables-functions-modeling
2. Number-Operations-Expressions-Equation

Take the number apart?

Tina, Emma, and Jen discuss this expression:

- $5 \frac{1}{3} \times 6$
- Tina: I know a way to multiply with a mixed number, like $5 \frac{1}{3}$, that is different from the one we learned in class. I call my way “take the number apart.” I’ll show you.

Which of the three girls do you think is right?
Justify your answer mathematically.

First, I multiply the 5 by the 6 and get 30.

Then I multiply the $\frac{1}{3}$ by the 6 and get 2. Finally, I add the 30 and the 2, which is 32.

- Tina: It works whenever I have to multiply a mixed number by a whole number.
- Emma: Sorry Tina, but that answer is wrong!
- Jen: No, Tina's answer is right for this one problem, but "take the number apart" doesn't work for other fraction problems.

What is an explanation?

Why you think it's true and why you think it makes sense.

Saying "distributive property" isn't enough, you have to show how the distributive property applies to the problem.

Example explanation

Why does $5 \frac{1}{3} \times 6 = (6 \times 5) + (6 \times \frac{1}{3})$?

Because

$$5 \frac{1}{3} = 5 + \frac{1}{3}$$

$$6(5 \frac{1}{3}) =$$

$$6(5 + \frac{1}{3}) =$$

$$(6 \times 5) + (6 \times \frac{1}{3}) \text{ because } a(b + c) = ab + ac$$

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Partial products

Place value in base 10

Factor $X^2 + 4x + 4$ in your head

Locate the difference, $p - m$, on the number line:



For each of the following cases, locate the quotient p/m on the number line :



Misconceptions about misconceptions

- They weren't listening when they were told
- They have been getting these kinds of problems wrong from day 1
- They forgot
- The other side in the math wars did this to the students on purpose

More misconceptions about the cause of misconceptions

- In the old days, students didn't make these mistakes
- They were taught procedures
- They were taught rich problems
- Not enough practice

Maybe

- Teachers' misconceptions perpetuated to another generation (where did the teachers get the misconceptions? How far back does this go?)
- Mile wide inch deep curriculum causes haste and waste
- Some concepts are hard to learn

Whatever the Cause

- When students reach your class they are not blank slates
- They are full of knowledge
- Their knowledge will be flawed and faulty, half baked and immature; but to them it is knowledge
- This prior knowledge is an asset and an interference to new learning

Second grade

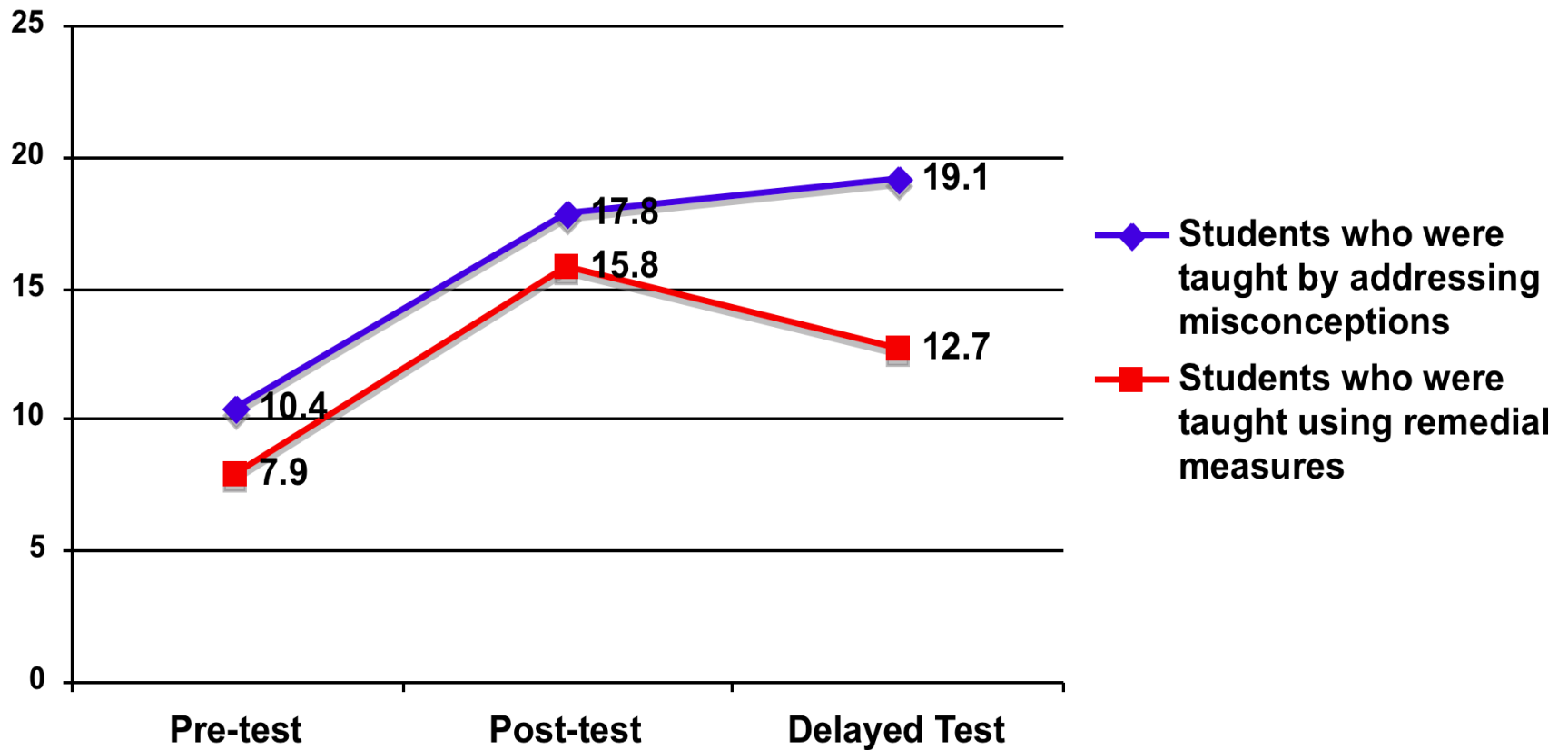
- When you add or subtract, line the numbers up on the right, like this:
- 23
- +9

- Not like this
- 23
- +9

Third Grade

- $3.24 + 2.1 = ?$
- If you “Line the numbers up on the right “ like you spent all last year learning, you get this:
- 3.24
- $+ 2.1$
- You get the wrong answer doing what you learned last year. You don't know why.
- Teach: line up decimal point.
- Continue developing place value concepts

Misconception Learning verses Remedial Learning: Test Scores



Lesson Units for Formative Assessment

- **Concept lessons**

“Proficient students expect mathematics to make sense”

- To reveal and develop students’ interpretations of significant mathematical ideas and how these connect to their other knowledge.

- **Problem solving lessons**

“They take an active stance in solving mathematical problems”

- To assess and develop students’ capacity to apply their Math flexibly to non-routine, unstructured problems, both from pure math and from the real world.

Mathematical Practices Standards

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College and Career Readiness Standards for Mathematics

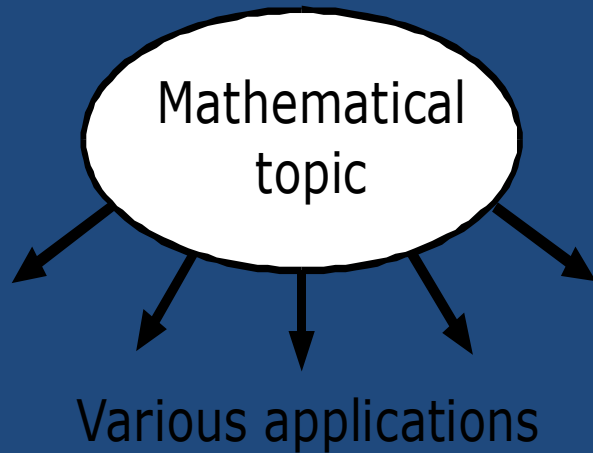
Mathematical Content Standards

- Number & Quantity
- Algebra
- Functions
- Modeling
- Statistics and Probability
- Geometry

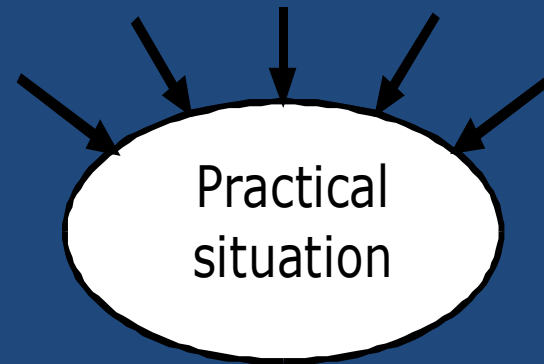
“Concept focused”

v

“Problem focused”:



Various mathematical tools



Why do students have to do math problems?

1. to get answers because Homeland Security needs them, pronto
2. I had to, why shouldn't they?
3. so they will listen in class
4. to learn mathematics

Why give students problems to solve?

To learn mathematics.

Answers are part of the process, they are not the product.

The product is the student's mathematical knowledge and know-how.

The 'correctness' of answers is also part of the process. Yes, an important part.

Wrong Answers

- Are part of the process, too
- What was the student thinking?
- Was it an error of haste or a stubborn misconception?

Three Responses to a Math Problem

1. Answer getting
2. Making sense of the problem situation
3. Making sense of the mathematics you can learn from working on the problem

Answers are a black hole: hard to escape the pull

- Answer getting short circuits mathematics, making mathematical sense
- Very habituated in US teachers versus Japanese teachers
- Devised methods for slowing down, postponing answer getting

Answer getting vs. learning mathematics

- USA:

How can I teach my kids to get the answer to this problem?

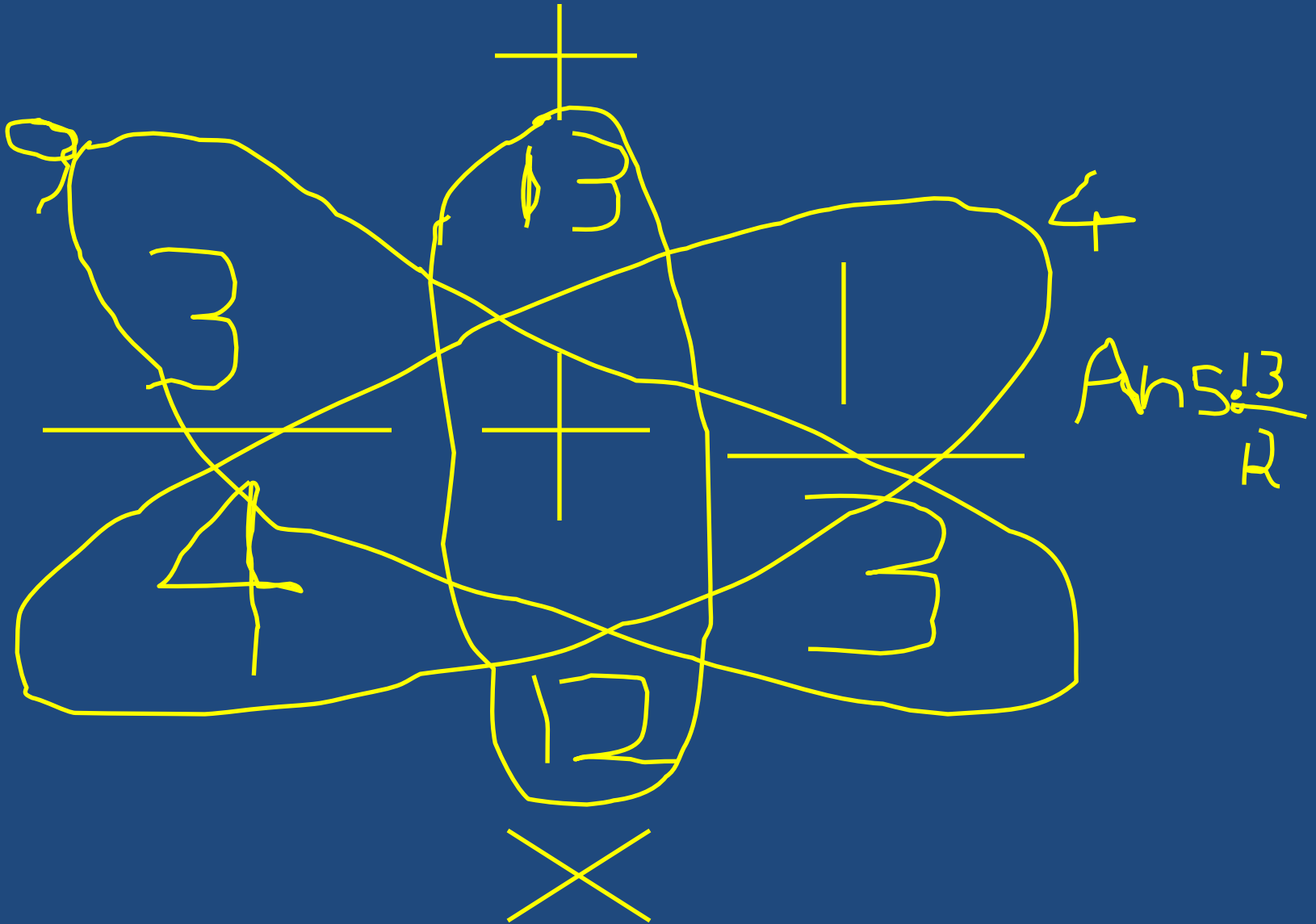
Use mathematics they already know. Easy, reliable, works with bottom half, good for classroom management.

- Japanese:

How can I use this problem to teach the mathematics of this unit?

Butterfly method

$$\frac{3}{4} + \frac{1}{3}$$



Use butterflies on this TIMSS item

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} =$$

Set up

- Not:
 - “set up a proportion and cross multiply”
- But:
 - Set up an equation and solve
- Prepare for algebra, not just next week’s quiz.

Foil FOIL

- Use the distributive property
- It works for trinomials and polynomials in general
- What is a polynomial?
- Sum of products = product of sums
- This IS the distributive property when “a” is a sum

Canceling

$$x^5/x^2 = x \cdot x \cdot x \cdot x \cdot x / x \cdot x$$

$$x^5/x^5 = x \cdot x \cdot x \cdot x \cdot x / x \cdot x \cdot x \cdot x \cdot x$$

Standards are a peculiar genre

1. We write as though students have learned approximately 100% of what is in preceding standards. This is never even approximately true anywhere in the world.
2. Variety among students in what they bring to each day's lesson is the condition of teaching, not a breakdown in the system. We need to teach accordingly.
3. Tools for teachers...instructional and assessment...should help them manage the variety

Differences among students

- The first response, in the classroom: make different ways of thinking students' bring to the lesson visible to all
- Use 3 or 4 different ways of thinking that students bring as starting points for paths to grade level mathematics target
- All students travel all paths: robust, clarifying

Social Justice

- Main motive for standards
- Get good curriculum to all students
- Start each unit with the variety of thinking and knowledge students bring to it
- Close each unit with on-grade learning in the cluster of standards

Differentiating lesson by lesson

Differentiating lesson by lesson:

- The arc of the lesson
- The structure of the lesson
- Using a problem to teach mathematics
- Classroom management and motivation
- Student thinking and closure

The arc of the lesson

- Diagnostic: make differences visible; what are the differences in mathematics that different students bring to the problem
- All understand the thinking of each: from least to most mathematically mature
- Converge on grade -level mathematics: pull students together through the differences in their thinking

Next lesson

- Start all over again
- Each day brings its differences, they never go away

Lesson Design

- Problem of the Day
- Lesson Opener
- Comprehensible Input/Modeling and Structured Practice
- Guided Practice
- Presentation (by student)
- Closure
- Preview

This design works well for introducing new procedural content to a group within range of the content

Adapted Lesson Structure

Adapted Lesson Structure for differentiating

- Pose problem whole class (3-5 min)
- Start work solo (1 min)
- Solve problem pair (10 min)
- Prepare to present pair (5 min)
- Selected S presents whole cls (15 min)
- Closure & Preview whole cls (5 min)

Posing the problem

- Whole class: pose problem, make sure students understand the language, no hints at solution
- Focus students on the problem situation, not the question/answer game. Hide question and ask them to formulate questions that make situation into a word problem
- Ask 3-6 questions about the same problem situation; ramp questions up toward key mathematics that transfers to other problems

What problem to use?

- Problems that draw thinking toward the mathematics you want to teach. NOT too routine, right after learning how to solve
- Ask about a chapter: what is the most important mathematics students should take with them? Find a problem that draws attention to this mathematics
- Begin chapter with this problem (from lesson 5 thru 10, or chapter test). This has diagnostic power. Also shows you where time has to go.
- Also Near end of chapter, while still time to respond

Solo-pair work

- Solo honors 'thinking' which is solo
- 1 minute is manageable for all, 2 minutes creates classroom management issues that aren't worth it.
- An unfinished problem has more mind on it than a solved problem
- Pairs maximize accountability: no place to hide
- Pairs optimize ear time: everyone is listened to
- You want divergence; diagnostic; make differences visible

Presentations

- All pairs prepare presentation
- Select 3-5 that show the spread, the differences in approaches from least to most mature
- Interact with presenters, engage whole class in questions
- Object and focus is for all to understand thinking of each, including approaches that didn't work; slow presenters down to make thinking explicit
- Go from least to most mature, draw with marker correspondences across approaches
- Converge on mathematical target of lesson

Close

- Use student presentations to illustrate and explain the key mathematical ideas of lesson
- Applaud
 - adaptive problem solving techniques that come up,
 - the dispositional behaviors you value,
 - the success in understanding each others thinking (name the thought)

The arc of a unit

- Early: diagnostic, organize to make differences visible
 - Pair like students to maximize differences between pairs
- Middle: spend time where diagnostic lessons show needs.
- Late: converge on target mathematics
 - Pair strong with weak students to minimize differences, maximize tutoring

Each lesson teaches the whole chapter

- Each lesson covers 3-4 weeks in 1-2 days
- Lessons build content by
 - increasing the resolution of details
 - Developing additional technical know-how
 - Generalizing range and complexity of problem situations
 - Fitting content into student reasoning
- This is not “spiraling”, this is depth and thoroughness for durable learning

Take the number apart?

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Inclusion, equity and social justice

- Standards should be within reach of the distribution of students.
- Focus so that there is time to be patient.
- Understanding thinking of others should be part of the standards, using the discipline's forms of discourse
- Pathways for students includes way for children to catch up.
- Standards that require less than the available time – teach less, learn more

The user has control

- Sometimes a tool is just right for the wrong use.

